Family Name:__________________**Given Name:**__________________ **I.D.#**_________

MAT3320 Assignment 2

Total: 10 marks. Due date: Tuesday, June 6, on or before 4:00pm.

In MATH Department (585 King Edward), there is a Drop-Box. You need to put your assignment into the box **on or before 4:00pm** on the due date. Late assignments will not be accepted.

1. (7 marks=1+1+2+3) Consider the following equation:

$$2xy'' + y' + 3y = 0.$$

(a) Show that $x_0 = 0$ is a regular singular point.

(b) Write down the indicial equation and solve it to determine r_1 and r_2, $r_1 \geq r_2$.

(c) Let $y = \sum_{n=0}^{\infty} c_n(r)x^{n+r}$. Determine the recursive relation for $c_n(r)$, i.e., relation between $c_{n+1}(r)$ and $c_n(r)$.

(d) Take $c_0(r) = 1$. Find two linearly independent solutions y_1 and y_2 which are valid for $x > 0$ near $x_0 = 0$.

Solution: (a) We rewrite the equation as

$$y'' + \frac{1}{2x}y' + \frac{3}{2x}y = 0.$$

$p(x) = \frac{1}{2x}$ and $q(x) = \frac{3}{2x}$ are not analytic at $x_0 = 0$; $xp(x) = \frac{1}{2}$ and $x^2q(x) = \frac{3}{2}x$ are analytic at $x_0 = 0$, so $x_0 = 0$ is a regular singular point.

(b) $p_0 = \frac{1}{2}$, $q_0 = 0$. Then we get the indicial equation: $r^2 - \frac{1}{2}r = 0$. Then $r_1 = \frac{1}{2}$, $r_2 = 0$. This is Case I.

(c) Substitute

$$y = \sum_{n=0}^{\infty} c_n(r)x^{n+r}, y' = \sum_{n=0}^{\infty}(n+r)c_n(r)x^{n+r-1}, y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n(r)x^{n+r-2}$$

into the differential equation to obtain

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n(r)x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n(r)x^{n+r-1} + \sum_{n=0}^{\infty} 3c_n(r)x^{n+r} = 0, \Rightarrow$$

$$\sum_{n=0}^{\infty}[(n+r)(2n+2r-1)c_n(r)x^{n+r-1} + \sum_{n=0}^{\infty} 3c_n(r)x^{n+r} = 0 \Rightarrow$$

$$r(2r-1)x^{r-1}+\sum_{n=1}^{\infty}[(n+r)(2n+2r-1)c_n(r)x^{n+r-1}+\sum_{n=0}^{\infty}3c_n(r)x^{n+r}=0 \Rightarrow$$

$$r(2r-1)x^{r-1}+\sum_{n=0}^{\infty}[(n+r+1)(2n+2r+1)c_{n+1}(r)x^{n+r}+\sum_{n=0}^{\infty}3c_n(r)x^{n+r}=0 \Rightarrow$$

$$r(2r-1)x^{r-1}+\sum_{n=0}^{\infty}[(n+r+1)(2n+2r+1)c_{n+1}(r)+3c_n(r)]x^{n+r}=0 \Rightarrow$$

$$(n+r+1)(2n+2r+1)c_{n+1}(r)+3c_n(r), \Rightarrow$$

$$c_{n+1}(r)=\frac{-3c_n(r)}{(n+r+1)(2n+2r+1)}, n \geq 0.$$

(d) Taking $r=r_1=\frac{1}{2}$. Then

$$c_{n+1}(\frac{1}{2})=\frac{-3}{(n+\frac{1}{2}+1)(2n+1+1)}c_n(\frac{1}{2})=\frac{-3}{(2n+3)(n+1)}c_n(\frac{1}{2}), \Rightarrow$$

$$c_n(\frac{1}{2})=\frac{(-1)^n 6^n}{(2n+1)!}.$$

Now we take $r=r_2=0$. Then

$$c_n(0)=\frac{(-1)^n 3^n}{(1)(2)\cdots(n)(1)(3)\cdots(2n-1)}=\frac{(-1)^n 3^n}{n!(2n-1)!!} \quad or \quad \frac{(-1)^n 6^n}{(2n)!}.$$

Thus

$$y_1=x^{1/2}\sum_{n=0}^{\infty}\frac{(-1)^n 6^n}{(2n+1)!}x^n, \quad y_2=\sum_{n=0}^{\infty}\frac{(-1)^n 6^n}{(2n)!}x^n.$$

2. (1 mark) Consider the differential equation
   $4x^2y''+4xy'+(3x-36)y=0, \quad x>0.$ (1)
   By letting $z=\sqrt{x}$, the equation can be changed to the following Bessel's equation

$$z^2y_z''+zy_z'+\left(3z^2-36\right)y=0, \quad z>0.$$

   Find the general solution of the equation (1).
   Solution: $\lambda^2=3$, and $\nu^2=9$, thus $\lambda=\sqrt{3}$, and $\nu=3$. The solution of this equation is

$$y=cJ_6(\sqrt{3}z)+dY_6(\sqrt{3}z)=cJ_6(\sqrt{3x})+dY_6(\sqrt{3x}), \quad c,d\in\mathbb{R}.$$

3. (2 marks) Consider the Sturm-Liouville problem

$$y'' + 4y' + \lambda y = 0, \quad 0 < x < \pi, \quad y'(0) = 0, y'(\pi) = 0.$$

Is $\lambda = 8$ an eigenvalue? If yes, find the corresponding eigenfunction; if no, explain why.

Solution: The indicial (characteristic) eqn is: $r^2 + 4r + 8 = 0$, $r = -2 \pm 2i$. Thus

$$y(x) = e^{-2x}(A\cos 2x + B\sin 2x) \Rightarrow$$

$$y'(x) = e^{-2x}(-2A\cos 2x - 2B\sin 2x - 2A\sin 2x + 2B\cos 2x) \Rightarrow$$

$$y'(0) = 0 = -2A + 2B, \Rightarrow A = B, \Rightarrow$$

$$y(x) = Ae^{-2x}(\cos 2x + \sin 2x) \quad \textit{and} \quad y'(x) = -4Ae^{-2x}\sin 2x.$$

By $y'(\pi) = 0$ we have $-4Ae^{-2\pi}\sin(2\pi) = 0$. Since $\sin(2\pi) = 0$, A can be any real number. Thus $\lambda = 8$ is an eigenvalue, the corresponding eigenfunctions are $y(x) = Ae^{-2x}(\cos 2x + \sin 2x)$, where $A \in \mathbb{R}$, and $A \neq 0$.